STRUCTURE OF THE CYCLE MAP FOR HILBERT SCHEMES OF FAMILIES OF NODAL CURVES

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ABSTRACT. We study the relative Hilbert scheme of a family of nodal (or smooth) curves, over a base of arbitrary dimension, via its (birational) *cycle map*, going to the relative symmetric product. We show the cycle map is the blowing up of the discriminant locus, which consists of cycles with multiple points. We study certain loci called *node scrolls* which play in important role in the geometry of the cycle map.

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In the classical (pre-1980) theory of (smooth) algebraic curves, a dominant role is played by divisors– equivalently, finite subschemes– and their parameter spaces, i.e. symmetric products. Notably, one of the first proofs of the existence of special divisors [3] was based on intersection theory on symmetric products. In more recent developments however, where the focus has been on moduli spaces of stable curves, subscheme methods have been largely absent, replaced by tools related to stable maps and their moduli spaces (see [8] for a sampling stressing Vakil's work). Our purpose in this paper (and others in this series) is to develop and apply global subscheme methods suitable for the study of stable curves and their families.

Fix a family of curves given by a flat projective morphism

$$\pi : X \to B$$

over an irreducible base, with fibres

$$X_b = \pi^{-1}(b), b \in B$$

which are nonsingular for the generic b and at worst nodal for every b. For example, X could be the universal family of automorphism-free curves over the appropriate open subset of $\overline{\mathcal{M}}_g$, the moduli space of Deligne-Mumford stable curves. Consider the *relative Hilbert scheme*

$$X_B^{[m]} = \mathrm{Hilb}_m(X/B),$$

which parametrizes length-m subschemes of X contained in fibres of π. This comes endowed with a *cycle map* (also called 'Hilb-to-Chow' map) to the relative symmetric product

$$c_m : X_B^{[m]} \to X_B^{(m)}.$$

Because $X_B^{(m)}$ may be considered 'elementary' (though it's highly singular- see [6]), c_m is a natural tool for studying $X_B^{[m]}$. The structure of c_m is the object of this paper. Our first main result is the following

Blowup Theorem. *c_m is equivalent to the blowing up of the discriminant locus*

$$D^m \subset X_B^{(m)},$$

which is the Weil divisor parametrizing nonreduced cycles.

In particular, we obtain an effective Cartier divisor

$$2\Gamma^{(m)} = c_m^{-1}(D^m)$$

so that $-2\Gamma^{(m)}$ can be identified with the natural $\mathcal{O}(1)$ polarization of the blowup. In fact, we shall see that $\Gamma^{(m)}$ also exists as Carter divisor, not necessarily effective, and the line bundle $\mathcal{O}(-\Gamma^{(m)})$, which is canonically defined, will be (abusively) called the

discriminant polarization (though 'half discriminant' is more accurate); we will also refer to $\Gamma^{(m)}$ itself sometimes the discriminant polarization. We emphasize that the Blowup Theorem is valid without dimension restrictions on B. As suggested by the Theorem, the discriminant polarization plays a central role in subsequent developments of geometry and intersection theory on the Hilbert schemes $X_B^{[m]}$.

The Blowup Theorem will be proven via an explicit construction, locally over the symmetric product $X_B^{(m)}$. This construction yields a wealth of information about the Hilbert scheme, in particular about its *singularity stratification*, which may be defined as follows. Let $\theta_1, ..., \theta_r$ be a collection of distinct relative nodes of the family, each living over its own boundary component, and let $n_1, ..., n_r$ be integers. Set

$$\mathcal{S}^{n.,m}(\theta_.; X/B) = \{z : c_m(z) \geq \sum n_i\theta_i\} \subset X_B^{[m]}$$

This is mainly interesting when all $n_i \geq 2$. In this case, we construct a surjection

$$\bigcup_{1\leq j_i\leq n_i-1, \forall 1\leq i\leq r} F_{j.}^{n.,m}(\theta.; X/B) \twoheadrightarrow S^{n.,m}(\theta_.; X/B)$$

where each $F_{j.}^{n.,m}(\theta.; X/B)$, called a *node polyscroll* (or node scroll, when $r = 1$), is a $(\mathbb{P}^1)^r$-bundle over the smaller Hilbert scheme $(X^{\theta.})^{[m-\sum n_i]}$, where $X^{\theta.}$ denotes the blowup (=partial normalization) of X in $\theta_1, ..., \theta_r$, defined over the intersection of the boundary components corresponding to the θ_i. The fibre parameter of each i-th factor the node polyscroll encodes a sort of higher-order ($n_i - 1$)-st order, in fact) slope locally at the i-th node, and these together constitute the additional information contained in the Hilbert scheme over the symmetric product.

Our second main result (see Theorems 7.7, 7.9) determines the structure of node polyscrolls as $(\mathbb{P}^1)^r$-bundles. In fact, the disjointness of the nodes implies that the $\mathbb{P}^1$ factors vary independently, which allows us to reduce to the case of node scrolls, i.e. $r = 1$.

Actually, what's essential for the enumerative theory of the Hilbert scheme, as studied e.g. in [5], and in which node scrolls play an essential role, is the structure of the node scroll F as *polarized* $\mathbb{P}^1$ bundle, that is, the rank-2 vector bundles E so that there is an isomorphism $\mathbb{P}(E) \simeq F$, under which the canonical $\mathcal{O}(1)$ polarization on $\mathbb{P}(E)$ associated to the projectivization corresponds to the restriction of the discriminant polarization $-\Gamma^{(m)}$ on F. To state the result (approximately), denote by θ_x, θ_y the node preimages on X^θ, and by ψ_x, ψ_y the relative cotangent spaces to X^θ/T along them, and by $[m-n]_*D$, for any divisor D on X^θ, the 'norm' of D, as divisor on $(X^\theta)^{[m-n]}$.

Node Scroll Theorem. *There is a polarized isomorphism*

$$F_j^n(\theta) = \mathbb{P}(\mathcal{O}(-D_j^n(\theta)) \oplus \mathcal{O}(-D_{j+1}^n(\theta)))$$

where

$$D_j^n(\theta) = -\binom{n-j+1}{2}\psi_x - \binom{j}{2}\psi_y + (n-j+1)[m-n]_*\theta_x + j[m-n]_*\theta_y + \Gamma^{[m-n]}.$$

This result, and its polyscroll analogue, reduce intersection theory on polyscroll to that of the Mumford tautological classes, about which a great deal is now known (see e.g. [8] and references therein). This will be expanded on in [5].

1. Set-up

Let

$$\pi : X \to B$$

be a flat family of nodal, generically smooth curves with X, B irreducible. Let $X^m_B, X^{(m)}_B$, respectively, denote the mth Cartesian and symmetric fibre products of X relative to B. Thus, there is a natural map

$$\omega_m : X^m_B \to X^{(m)}_B$$

which realizes its target as the quotient of its source under the permutation action of the symmetric group $\mathfrak{S}_n$. Let

$$\mathrm{Hilb}_m(X/B) = X^{[m]}_B$$

denote the relative Hilbert scheme paramerizing length-m subschemes of fibres of π, and

$$c = c_m : X^{[m]}_B \to X^{(m)}_B$$

the natural *cycle map* (cf. [1]). Let $D^m \subset X^{(m)}_B$ denote the discriminant locus or 'big diagonal', consisting of cycles supported on $< m$ points (endowed with the reduced scheme structure). Clearly, D^m is a prime Weil divisor on $X^{(m)}_B$, birational to $X \times_B \mathrm{Sym}^{m-2}(X/B)$ (though it is less clear what the defining equations of D^m on $X^{(m)}_B$ are near singular points). The main result of Sections 1-4 is the

Theorem 1.1 (Blowup Theorem)**.** *The cycle map*

$$c_m : X^{[m]}_B \to X^{(m)}_B$$

is equivalent to the blowing up of $D^m \subset X^{(m)}_B$.

2. Preliminary reductions

To begin with, we reduce the Theorem to a local statement over a neighborhood of a 1-point cycle $mp \in X^{(m)}_B$ where $p \in X$ is a node of $\pi^{-1}(\pi(p))$. It was shown in [6], and will be reviewed below, that c_m is a small birational map (with fibres of dimension $\leq \min(m/2, \max\{|\mathrm{sing}(X_b)|, b \in B\})$), all its fibres (aka punctual Hilbert schemes or products thereof) are reduced, and that $X^{[m]}_B$ is reduced and irreducible (given that X is). Therefore the claimed equivalence is locally uniquely determined if it exists.

2.1. **Reduction 1: partly singular.** Now, there is a natural correspondence

$$G = X_B^{[m]} \times_{X_B^{(m)}} B_{D^m}(X_B^{(m)})$$

between the Hilbert scheme and the blowup, and Theorem 1.1 is precisely the statement that the maps

$$G \to X_B^{[m]}, G \to B_{D^m}(X_B^{(m)}),$$

which are a priori birational, are both isomorphisms or equivalently, étale. This statement is obviously local over $X_B^{(m)}$. Let $U \subset X_B^{(m)}$ denote the open subset consisting of cycles having multiplicity at most 1 at each fibre node. Then the cycle map $c_m : c_m^{-1}(U) \to U$ is an isomorphism. Besides, $D^m \cap U$ is Cartier, so the blowup map is an isomorphism there. Therefore the Theorem is certainly true over U. Consequently, it will suffice to show c_m is equivalent to the blowing-up of D^m locally near any cycle $Z \in X_B^{(m)}$ having degree > 1 at some point of the locus $X^\sigma \subset X$ of singular points of π (i.e. the union of all fibre nodes).

Remark 2.1. In fact, $G \to X_B^{[m]}$ is a priori an isomorphism, i.e. c_m factors through $B_{D^m}(X_B^{(m)})$. In other words (by the universal property of blowong up), $c_m^{-1}(D^m)$ is a Cartier divisor. This assertion is true because

(i) is is local over $X_B^{(m)}$ and compatible with base-change;
(ii) therefore is is sufficient to prove it for the 'standard model' H_m (see below) which is smooth;
(iii) in H_m, $c_m^{-1}(D^m)$ has no embedded points (see Lemma 4.5 below), hence it is Cartier.

2.2. **Reduction 2: maximally singular.** Reducing further, a rather standard 'splitting' argument that now describe shows that it will suffice to analyze c_m locally over a neighborhood of a 'maximally singular' fibre, i.e. one of the form mp where p is a fibre node (which is locally analytically a cross-section of π). For a cycle $Z \in X_B^{(m)}$, we let $(X_B^{(m)})_{(Z)}$ denote its open neighborhood consisting of cycles Z' which are 'no worse' than Z, in the sense that their support $\mathrm{supp}(Z')$ has degree (over B) at least equal to that of $\mathrm{supp}(Z)$. Similarly, for a k-tuple $Z. = (Z_1, ..., Z_k) \in \prod X_B^{(m_i)}$, we denote by $(\prod X_B^{(m_i)})_{(Z.)}$ its open neighborhood in the product consisting of 'no worse' multicycles, i.e. k-tuples $Z'. \in \prod X_B^{(m_i)}$ such that each Z'_i is no worse than Z_i and the various Z'_i are mutually pairwise disjoint. We also denote by $(X_B^{[m]})_{(Z)}, (\prod X_B^{[m_i]})_{(Z.)}$ the respective preimages of the no-worse neighborhoods of Z and $Z.$ via c_m and $\prod c_{m_i}$. Also, for a map $Y \xrightarrow{f} X_B^{(m)}$, we write $Y_{(Z)}$ for $f^{-1}((X_B^{(m)})_{(Z)})$.

Now writing a general cycle

$$Z = \sum_{i=1}^{k} m_i p_i$$

with $m_i > 0, p_i$ distinct, and setting $Z_i = m_i p_i$, we have a cartesian (in each square) diagram

$$\begin{array}{ccc} (\prod_{i=1}^{k}{}_B\, X_B^{[m_i]})_{(Z.)} & \xrightarrow{\prod c_{m_i}} & (\prod_{i=1}^{k}{}_B\, X_B^{(m_i)})_{(Z.)} \\ e_1 \uparrow & \square & \uparrow d_1 \\ H & \to & S \\ e \downarrow & \square & \downarrow d \\ (X_B^{[m]})_{(Z)} & \xrightarrow{c_m} & (X_B^{(m)})_{(Z)} \end{array}$$

Here H is the restriction of the natural inclusion correspondence on Hilbert schemes:

$$H = \{(\zeta_1, ..., \zeta_k, \zeta) \in (\prod_{i=1}^{k}{}_B\, X_B^{[m_i]})_{(Z.)} \times (X_B^{[m]})_{(Z)} : \zeta_i \subseteq \zeta, i = 1, ..., k\},$$

and similarly for S. Note that the right vertical arrows d, d_1 are étale and induce analytic isomorphisms between some analytic neighborhoods U of Z and U' of $Z.$ and the left vertical arrows e, e_1 are also étale and induce isomorphisms between $c_m^{-1}(U)$ and $(\prod c_{m_i})^{-1}(U')$.

Now by definition, the blow-up of $X_B^{(m)}$ in D^m is the Proj of the graded algebra

$$A(\mathcal{I}_{D^m}) = \bigoplus_{n=0}^{\infty} \mathcal{I}_{D^m}^n.$$

Note that

$$d^{-1}(D^m) = \sum p_i^{-1}(D^{m_i})$$

and moreover,

$$d^*(\mathcal{I}_{D^m}) = \bigotimes{}_B\, p_i^*(\mathcal{I}_{D^{m_i}})$$

where we use p_i generically to denote an ith coordinate projection. Therefore,

$$A(\mathcal{I}_{D^m}) \simeq \bigotimes{}_B\, p_i^* A(\mathcal{I}_{D^{m_i}})$$

as graded algebras, compatibly with the isomorphism

$$\mathcal{O}_{\prod_{i=1}^{k}{}_B\, \mathrm{Sym}^{m_i}(X/B)} \simeq \bigotimes_{i=1}^{k}{}_B\, \mathcal{O}_{\mathrm{Sym}^{m_i}(X/B)}.$$

Now it is a general fact that Proj is compatible with tensor product of graded algebras, in the sense that

$$\mathrm{Proj}(\bigotimes{}_B\, A_i) \simeq \prod{}_B\, \mathrm{Proj}(A_i).$$

Therefore there is an analogous correspondence

$$(\prod_{i=1}^{k}{}_B\, B_{D^{m_i}} X_B^{(m_i)})_{(Z.)} \leftarrow \tilde{S} \to (B_{D^m} X_B^{(m)})_{(Z)}$$

Consequently (1.2.2) induces another cartesian diagram

$$
\begin{array}{ccccc}
(\prod\limits_{i=1}^{k}{}_B\, X_B^{[m_i]})_{(Z.)} & \leftarrow & \prod G_i & \xrightarrow{\prod c'_{m_i}} & (\prod\limits_{i=1}^{k}{}_B\, B_{D^{m_i}} X_B^{(m_i)})_{(Z.)} \\
\uparrow & \square & \uparrow & \square & \uparrow \\
H & \leftarrow & \tilde{G} & \rightarrow & \tilde{S} \\
\downarrow & \square & \downarrow & \square & \downarrow \\
(X_B^{[m]})_{(Z)} & \leftarrow & G & \xrightarrow{c'_m} & (B_{D^m} X_B^{(m)})_{(Z)}.
\end{array}
\tag{2.2.1}
$$

The vertical arrows of (2.2.1) are all étale. Therefore, to prove the bottom arrows are étale it suffices to prove that the top arrows are, which would follow if we can prove étaleness for each G_i factor.

The upshot of all this is that it suffices to prove $c = c_m$ is equivalent to the blow-up of $X_B^{(m)}$ in D^m, locally over a neighborhood of a cycle of the form $mp, p \in X$, and we may obviously assume that p is a fibre node of π. One way to obtain such a neighborhood is to consider $U_B^{(m)}$ where U is a neighborhood of p in X.

2.3. **Reduction 3: standard coordinate neighborhood.** At this point, we may replace X by an analytic or formal (or étale) neighborhood U of p in X. Typically, we will take for U a tubular neighborhood of a relative node θ of X/B extending p, i.e. a branch of the singular locus of X/B through p. In a suitable such neighborhood (étale if p is a separating node, analytic in general), the family U/B is given by pulling back a universal local deformation

$$t = xy. \tag{2.3.1}$$

We call such U a *standard coordinate neighborhood*. Since both the relative Hilbert scheme and the blowing-up process are compatible with pullback, we may as well– as long as it's just a matter of proving Theorem 1.1– assume that X/B is itself given by (2.3.1). Since our interest is also in global identifications however, we shall be making this assumption only part of the time.

3. A LOCAL MODEL

We now reach the heart of the matter: an explicit construction, locally over the symmetric product, of the relative Hilbert scheme in terms of coordinates. This construction will have many applications beyond the proof of the Blowup Theorem. We begin with some preliminaries.

3.1. **Symmtric product.** Assuming the local form (2.3.1), then the relative cartesian product U_B^m, as subscheme of $U^m \times B$, is given locally by

$$x_1y_1 = ... = x_my_m = t.$$

Let $\sigma_i^x, \sigma_i^y, i = 0, ..., m$ denote the elementary symmetric functions in $x_1, ..., x_m$ and in $y_1, ..., y_m$, respectively, where we set $\sigma_0 = 1$. We note that these functions satisfy the relations

$$\sigma_m^y \sigma_j^x = t^j \sigma_{m-j}^y, \ \ \sigma_m^x \sigma_j^y = t^j \sigma_{m-j}^x, \tag{3.1.1}$$

$$t^{m-i}\sigma_{m-j}^y = t^{m-i-j}\sigma_j^x \sigma_m^y, \ \ t^{m-i}\sigma_{m-j}^x = t^{m-i-j}\sigma_j^y \sigma_m^x \tag{3.1.2}$$

(of course the second set are consequences of the first). Putting the sigma functions together with the projection to B, we get a map

$$\sigma : \mathrm{Sym}^m(U/B) \to \mathbb{A}_B^{2m} = \mathbb{A}^{2m} \times B$$

$$\sigma = ((-1)^m \sigma_m^x, ..., -\sigma_1^x, (-1)^m \sigma_m^y, ..., -\sigma_1^y, \pi^{(m)})$$

where $\pi^{(m)} : X_B^{(m)} \to B$ is the structure map.

Lemma 3.1. *σ is an embedding locally near mp.*

Proof. It suffices to prove this formally, i.e. to show that $\sigma_i^x, \sigma_j^y, i, j = 1, ..., m$ generate topologically the completion $\hat{\mathfrak{m}}$ of the maximal ideal of mp in $X_B^{(m)}$. To this end it suffices to show that any $\mathfrak{S}_m$-invariant polynomial in the x_i, y_j is a polynomial in the σ_i^x, σ_j^y and t. Let us denote by R the averaging or symmetrization operator with respect to the permutation action of $\mathfrak{S}_m$, i.e.

$$R(f) = \frac{1}{m!} \sum_{g \in \mathfrak{S}_m} g^*(f).$$

Then it suffices to show that the elements $R(x^I y^J)$, where x^I (resp. y^J) range over all monomials in $x_1, ..., x_m$ (resp. $y_1, ..., y_m$) are polynomials in the σ_i^x, σ_j^y and t. Because $x_i y_i = t$, we may assume $I \cap J = \emptyset$. On the other hand, expanding the product $R(x^I)R(y^J)$ we get a sum of monomials $x^{I'}y^{J'}$ times a rational number; those with $I' \cap J' = \emptyset$ add up to $\frac{1}{m!}R(x^I y^J)$, while those with $I' \cap J' \neq \emptyset$ are divisible by t. Thus,

$$R(x^I y^J) - m! R(x^I) R(y^J) = tF$$

where F is an $\mathfrak{S}_m$-invariant polynomial in the x_i, y_j of bidegree $(|I| - 1, |J| - 1)$, hence a linear combination of elements of the form $R(x^{I'}y^{J'}), |I'| = |I| - 1, |J'| = |J| - 1$. By induction, F is a polynomial in the σ_i^x, σ_j^y and clearly so is $R(x^I)R(y^J)$. Hence so is $R(x^I y^J)$ and we are done. $\square$

Remark 3.2. It will follow from Theorem 1 and its proof that the equations (3.1.1-3.1.2) actually define the image of σ scheme-theoretically (see Cor. 4.3 below); we won't need this, however.

3.2. **A projective family.** Now we present a construction of our local model $\tilde{H}$. This is motivated by our study in [7] of the relative Hilbert scheme of a node. As we saw there, the fibres of the cycle map are chains consisting of n rational curves where n takes the values from $n = 0$ for the generic fibre (meaning the fibre is a singleton) to $n = m - 1$ for the most special fibre. Therefore, it is reasonable to try model the cycle map on a standard pencil of rational normal $(m-1)$-tics specializing to a chain of lines. Further motivation for the construction that follows comes from [6], where an explicit construction is given for the full-flag Hilbert scheme.

Let $C_1, ..., C_{m-1}$ be copies of $\mathbb{P}^1$, with homogenous coordinates u_i, v_i on the i-th copy. Let

$$\tilde{C} \subset C_1 \times ... \times C_{m-1} \times B/B$$

be the subscheme over B defined by

$$v_1 u_2 = t u_1 v_2, ..., v_{m-2} u_{m-1} = t u_{m-2} v_{m-1}. \tag{3.2.1}$$

This construction is motivated (cf. [6]) by viewing u_i/v_i as a stand-in for x_{I^c}/y_I where $I \subset [1, m]$ is any i-tuple; the ratio is independent of I. In any event, $\tilde{C}$ is a reduced complete intersection of divisors of type $(1, 1, 0, ..., 0), (0, 1, 1, 0, ..., 0)$,..., $(0, ..., 0, 1, 1)$ (relatively over B) and it is easy to check that the fibre of $\tilde{C}$ over $0 \in B$ is

$$\tilde{C}_0 = \bigcup_{i=1}^{m-1} \tilde{C}_i, \tag{3.2.2}$$

where

$$\tilde{C}_i = [1, 0] \times ... \times [1, 0] \times C_i \times [0, 1] \times ... \times [0, 1]$$

and that in a neighborhood of the special fibre $\tilde{C}_0$, $\tilde{C}$ is smooth and $\tilde{C}_0$ is its unique singular fibre over B. We may embed $\tilde{C}$ in $\mathbb{P}^{m-1} \times B$, relatively over B using the plurihomogenous monomials

$$Z_i = u_1 \cdots u_{i-1} v_i \cdots v_{m-1}, i = 1, ..., m. \tag{3.2.3}$$

These satisfy the relations

$$Z_i Z_j = t^{j-i-1} Z_{i+1} Z_{j-1}, i < j - 1 \tag{3.2.4}$$

so they embed $\tilde{C}$ as a family of rational normal curves $\tilde{C}_t \subset \mathbb{P}^{m-1}, t \neq 0$ specializing to $\tilde{C}_0$, which is embedded as a nondegenerate, connected chain of $m - 1$ lines.

3.3. **To Hilb.** Next consider an affine space $\mathbb{A}^{2m}$ with coordinates $a_0, ..., a_{m-1}, d_0, ..., d_{m-1}$. The a_i, d_j are to play the roles of $\sigma^x_{m-i}, \sigma^y_{m-j}$ respectively (where as we recall u_i/v_i plays that of $x_1...x_{m-i}/y_{m-i+1}...y_m$). With this and the relations (3.1.1), (3.1.2) in mind, let $\tilde{H} \subset \tilde{C} \times \mathbb{A}^{2m}$ be the subscheme defined by

$$\begin{gathered} a_0 u_1 = t v_1, d_0 v_{m-1} = t u_{m-1} \\ a_1 u_1 = d_{m-1} v_1, ..., a_{m-1} u_{m-1} = d_1 v_{m-1}. \end{gathered} \tag{3.3.1}$$

Set $L_i = p_{C_i}^* \mathcal{O}(1)$. Then consider the subscheme of $Y = \tilde{H} \times_B U$ defined by the equations

$$F_0 := x^m + a_{m-1}x^{m-1} + \dots + a_1 x + a_0 \in \Gamma(Y, \mathcal{O}_Y) \tag{3.3.2}$$

$$F_1 := u_1 x^{m-1} + u_1 a_{m-1} x^{m-2} + \dots + u_1 a_2 x + u_1 a_1 + v_1 y \in \Gamma(Y, L_1) \tag{3.3.3}$$

$$\dots$$

$$F_i := u_i x^{m-i} + u_i a_{m-1} x^{m-i-1} + \dots + u_i a_{i+1} x + u_i a_i + v_i d_{m-i+1} y + \dots + v_i d_{m-1} y^{i-1} + v_i y^i$$
$$\in \Gamma(Y, L_i) \tag{3.3.4}$$

$$\dots$$

$$F_m := d_0 + d_1 y_1 + \dots + d_{m-1} y^{m-1} + y^m \in \Gamma(Y, \mathcal{O}_Y). \tag{3.3.5}$$

The following statement summarizes results from [7].

Theorem 3.3. (i) *$\tilde{H}$ is smooth and irreducible.*

(ii) *The ideal sheaf $\mathcal{I}$ generated by $F_0, \dots, F_m$ defines a subscheme of $\tilde{H} \times_B X$ that is flat of length m over $\tilde{H}$*

(iii) *The classifying map*

$$\Phi = \Phi_{\mathcal{I}} : \tilde{H} \to Hilb_m(U/B)$$

is an isomorphism.

(iv) *Φ induces an isomorphism*

$$(\tilde{C})_0 = p_{\mathbb{A}^{2m}}^{-1}(0) \to Hilb_m^0(X_0) = \bigcup_{i=1}^{m} C_i^m$$

(cf. [7]) of the fibre of $\tilde{H}$ over $0 \in \mathbb{A}^{2m}$ with the punctual Hilbert scheme of the special fibre X_0, in such a way that the point $[u, v] \in \tilde{C}_i \sim C_i \sim \mathbb{P}^1$ corresponds to

- *the subscheme $I_i^m(u/v) = (x^{m-i} + (u/v)y^i) \in C_i^m \subset Hilb_m^0(X_0)$ if $uv \neq 0$,*
- *the subscheme $(x^{m+1-i}, y^i) \in C_i^m$ if $[u, v] = [0, 1]$,*
- *the subscheme $(x^{m-i}, y^{i+1}) \in C_i^m$ if $[u, v] = [1, 0]$.*

(v) *over U_i, a co-basis for the universal ideal $\mathcal{I}$ (i.e. a basis for $\mathcal{O}/\mathcal{I}$) is given by*

$$1, \dots, x^{m-i}, y, \dots, y^{i-1}.$$

(vi) *Φ induces an isomorphism of the special fibre $\tilde{H}_0$ of H over B with $Hilb_m(X_0)$, and $\tilde{H}_0 \subset \tilde{H}$ is a divisor with global normal crossings $\bigcup_{i=0}^{m} D_i^m$ where each D_i^m is smooth, birational to $(x-axis)^{m-i} \times (y-axis)^i$, and has special fibre C_i^m under the cycle map $p_{\mathbb{A}^m}$.*

Proof. The smoothness of $\tilde{H}$ is clear from the defining equations equations and also follows from smoothness of $\mathrm{Hilb}_m(U/B)$ once (ii) and (iii) are proven. To that end consider the point $q_i, i = 1, \dots, m$, on the special fibre of $\tilde{H}$ over $\mathbb{A}_B^{2m}$ with coordinates

$$v_j = 0,\ \forall j < i; u_j = 0,\ \forall j \geq i.$$

Then q_i has an affine neighborhood U_i in $\tilde{H}$ defined by

$$U_i = \{u_j = 1,\ \forall j < i;\ v_j = 1,\ \forall j \geq i\},$$

and these $U_i, i = 1, ..., m$ cover a neighborhood of the special fibre of $\tilde{H}$. Now the generators F_i admit the following relations:

$$u_{i-1}F_j = u_j x^{i-1-j} F_{i-1},\ 0 \leq j < i-1;\ v_i F_j = v_j y^{j-i} F_i,\ m \geq j > i$$

where we set $u_i = v_i = 1$ for $i = 0, m$. Hence $\mathcal{I}$ is generated there by F_{i-1}, F_i and assertions (ii), (iii) follow directly from Theorems 1,2 and 3 of [7] and (iv) is obvious.

As for (v), it follows immediately from the definition of the F_i, plus the fact just noted that, over U_i, the ideal $\mathcal{I}$ is generated by F_{i-1}, F_i, and that on U_i, we have $u_{i-1} = v_i = 1$. Finally (vi) is contained in [7], Thm. 2.

□

In light of Theorem 3.3, we identify a neighborhood H_m of the special fibre in $\tilde{H}$ with a neighborhood of the punctual Hilbert scheme (i.e. $c_m^{-1}(mp)$) in $X_B^{[m]}$, and note that the projection $H_m \to \mathbb{A}^{2m} \times B$ coincides generically, hence everywhere, with $\sigma \circ c_m$. Hence H_m may be viewed as the subscheme of $\mathrm{Sym}^m(U/B) \times_B \tilde{C}$ defined by the equations

$$\begin{gathered} \sigma_m^x u_1 = t v_1, \\ \sigma_{m-1}^x u_1 = \sigma_1^y v_1, ..., \sigma_1^x u_{m-1} = \sigma_{m-1}^y v_{m-1}, \\ t u_{m-1} = \sigma_m^y v_{m-1} \end{gathered} \tag{3.3.6}$$

Alternatively, in terms of the Z coordinates, H_m may be defined as the subscheme of $\mathrm{Sym}^m(U/B) \times \mathbb{P}_Z^{m-1} \times B$ defined by the relations (3.2.4), which define $\tilde{C}$, together with

$$\sigma_i^y Z_i = \sigma_{m-i}^x Z_{i+1},\ i = 1, ..., m-1 \tag{3.3.7}$$

4. Reverse engineering and proof of Blowup Theorem

Reverse-engineering an ideal means finding generators with given syzygies. Our task now is effectively to reverse-engineer an ideal (discriminant ideal) in the σ's whose syzygies (for suitable generators) are given by (3.3.7) and (3.2.4). This will be achieved by passing to the ordered version of Hilb. The sought-for generators will be given by certain 'mixed Van der Monde' determinants. The proof of the Blowup Theorem is then concluded, essentially by showing explicitly that, locally over Hilb, all the generators are multiples of one of them.

4.1. **Order.** Let $OH_m = H_m \times_{\mathrm{Sym}^m(U/B)} U_B^m$, so we have a cartesian diagram

$$\begin{array}{ccc} OH_m & \xrightarrow{\varpi_m} & H_m \\ oc_m \downarrow & \square & \downarrow c_m \\ X_B^m & \xrightarrow{\omega_m} & X_B^{(m)} \end{array}$$

and its global analogue

$$\begin{array}{ccc} X_B^{[m]} & \xrightarrow{\varpi_m} & X_B^{[m]} \\ oc_m \downarrow & \square & \downarrow c_m \\ X_B^m & \xrightarrow{\omega_m} & X_B^{(m)} \end{array}$$

Note that $X_B^{(m)}$ is normal and Cohen-Macaulay: this follows from the fact that it is a quotient by $\mathfrak{S}_m$ of X_B^m, which is a locally complete intersection with singular locus of codimension ≥ 2 (in fact, > 2, since X is smooth). Alternatively, normality of $X_B^{(m)}$ follows from the fact that H_m is smooth and the fibres of
$c_m : H_m \to X_B^{(m)}$ are connected (being products of connected chains of rational curves). Note that ω_m is simply ramified generically over D^m and we have

$$\omega_m^*(D^m) = 2OD^m$$

where

$$OD^m = \sum_{i<j} D_{i,j}^m$$

where $D_{i,j}^m = p_{i,j}^{-1}(OD^2)$ is the locus of points whose ith and jth components coincide. To prove c_m is equivalent to the blowing-up of D^m it will suffice to prove that oc_m is equivalent to the blowing-up of $2OD^m = \omega_m^*(D^m)$ which in turn is equivalent to the blowing-up of OD^m. The advantage of working with OD^m rather than its unordered analogue is that at least some of its equations are easy to write down: let

$$v_x^m = \prod_{1 \leq i < j \leq m} (x_i - x_j),$$

and likewise for v_y^m. As is well known, v_x^m is the determinant of the Van der Monde matrix

$$V_x^m = \begin{bmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_m \\ \vdots & & \vdots \\ x_1^{m-1} & \dots & x_m^{m-1} \end{bmatrix}.$$

Also set

$$\tilde{U}_i = \varpi_m^{-1}(U_i),$$

where U_i is as in (1.3.7), being a neighborhood of q_i on H_m. Then in U_1, the universal ideal $\mathcal{I}$ is defined by

$$F_0, \;\; F_1 = y + (\text{function of } x)$$

and consequently the length-m scheme corresponding to $\mathcal{I}$ maps isomorphically to its projection to the x-axis. Therefore over $\tilde{U}_1 = \varpi_m^{-1}(U_1)$, where F_0 splits as $\prod(x - x_i)$, the equation of OD^m is simply given by

$$G_1 = v_x^m.$$

Similarly, the equation of OD^m in $\tilde{U}_m$ is given by

$$G_m = v_y^m.$$

New let

$$\Xi : OH_m \to \mathbb{P}^{m-1}$$

be the morphism corresponding to $[Z_1, ..., Z_m]$, and set

$$L = \Xi^*(\mathcal{O}(1)). \tag{4.1.1}$$

Note that $\tilde{U}_i$ coincides with the open set where $Z_i \neq 0$, so Z_i generates L over $\tilde{U}_i$. Let

$$O\Gamma^{(m)} = oc_m^{-1}(OD^m).$$

We shall see below that this is a Cartier divisor, in fact we shall construct an isomorphism

$$\gamma : \mathcal{O}(-O\Gamma^{(m)}) \to L. \tag{4.1.2}$$

This isomorphism will easily imply Theorem 1. To construct γ, it suffices to specify it on each $\tilde{U}_i$.

4.2. **Mixed Van der Mondes.** A clue as to how the latter might be done comes from the relations (1.4.2-1.4.3). Thus, set

$$G_i = \frac{(\sigma_m^y)^{i-1}}{t^{(i-1)(m-i/2)}} v_x^m = \frac{(\sigma_m^y)^{i-1}}{t^{(i-1)(m-i/2)}} G_1, \ \ i = 2, ..., m. \tag{4.2.1}$$

Thus,

$$G_2 = \frac{\sigma_m^y}{t^{m-1}} G_1, G_3 = \frac{\sigma_m^y}{t^{m-2}} G_2, ..., G_{i+1} = \frac{\sigma_m^y}{t^{m-i}} G_i, i = 1, ..., m-1. \tag{4.2.2}$$

In light of (3.1.1),(3.1.2), we deduce

$$\sigma_{m-i}^x G_{i+1} = \sigma_i^y G_i. \tag{4.2.3}$$

Comparing this with (3.3.7) certainly suggests solving our reverse-engineering problem by assigning Z_i to G_i, which is what we will do eventually.

Now an elementary calculation shows that if we denote by V_i^m the 'mixed Van der Monde' matrix

$$V_i^m = \begin{bmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_m \\ \vdots & & \vdots \\ x_1^{m-i} & \dots & x_m^{m-i} \\ y_1 & \dots & y_m \\ \vdots & & \vdots \\ y_1^{i-1} & \dots & y_m^{i-1} \end{bmatrix}$$

then we have

$$G_i = \pm \det(V_i^m), i = 1, ..., m. \tag{4.2.4}$$

⌜ Indeed for $i = 1$ this is standard; for general i, it suffices to prove the analogue of (4.2.2) for the mixed Van der Monde determinants. For this, it suffices to multiply each jth column of V_i^m by y_j, and factor a $t = x_j y_j$ out of each of rows $2, ..., m-i+1$, which yields

$$\sigma_m^y \det(V_i^m) = (-1)^{m-i+1} t^{m-i} V_{i+1}^m. \tag{4.2.5}$$

⌞ From (4.2.4) it follows, e.g., that G_m as given in (4.2.1) coincides with v_y^m.

4.3. **Conclusion of proof.** The following result is key for the Blowup Theorem.

Lemma 4.1. *G_i generates $\mathcal{O}(-O\Gamma^{(m)})$ over $\tilde{U}_i$. In particular, $O\Gamma^{(m)}$ is Cartier.*

Proof of Lemma. This is clearly true where $t \neq 0$ and it remains to check it along the special fibre $OH_{m,0}$ of OH_m over B. Note that $OH_{m,0}$ is a sum of components of the form

$$\Theta_I = \mathrm{Zeros}(x_i, i \notin I, y_i, i \in I), I \subseteq \{1, ..., m\}, \tag{4.3.1}$$

none of which is contained in the singular locus of OH_m. Set

$$\Theta_i = \bigcup_{|I|=i} \Theta_I.$$

Note that

$$\tilde{C}_i \times 0 \subset \Theta_i, i = 1, ..., m-1$$

and therefore

$$\tilde{U}_i \cap \Theta_j = \emptyset, j \neq i-1, i.$$

Note that y_i vanishes to order 1 (resp. 0) on Θ_I whenever $i \in I$ (resp. $i \notin I$). Similarly, $x_i - x_j$ vanishes to order 1 (resp. 0) on Θ_I whenever both $i, j \in I^c$ (resp. not both $i, j \in I^c$). From this, an elementary calculation shows that the vanishing order of G_j on every component Θ of Θ_k is

$$\mathrm{ord}_\Theta(G_j) = (k-j)^2 + (k-j). \tag{4.3.2}$$

We may unambiguously denote this number by $\mathrm{ord}_{\Theta_k}(G_j)$. Since this order is nonnegative for all k, j, it follows firstly that the rational function G_j has no poles, hence is in fact regular on X_B^m near mp (recall that X_B^m is normal); of course, regularity of G_j is also immediate from (4.2.4). Secondly, since this order is zero for $k = j, j-1$, and Θ_j, Θ_{j-1} contain all the components of $OH_{m,0}$ meeting $\tilde{U}_j$, it follows that in $\tilde{U}_j$, G_j has no zeros besides $O\Gamma^{(m)} \cap \tilde{U}_j$, so G_j is a generator of $\mathcal{O}(-O\Gamma^{(m)})$ over $\tilde{U}_j$. QED Lemma. □

The Lemma yields a set of generators for the ideal of OD^m:

Corollary 4.2 (of Lemma). *The ideal of OD^m is generated, locally near p^m, by $G_1, ..., G_m$.*

Proof. If Q denotes the cokernel of the map $m\mathcal{O}_{X^m} \to \mathcal{O}_{X^m}(-OD^m)$ given by $G_1, ..., G_m$, then $c_m^*(Q) = 0$ by the Lemma, hence $Q = 0$, so the G's generate $\mathcal{O}_{X^m}(-OD^m)$. □

Now we can construct the desired isomorphism γ as in (4.1.2), as follows. Since Z_j is a generator of L over $\tilde{U}_j$, we can define our isomorphism γ over $\tilde{U}_j$ simply by specifying that

$$\gamma(G_j) = Z_j \text{ on } \tilde{U}_j.$$

Now to check that these maps are compatible, it suffices to check that

$$G_j/G_k = Z_j/Z_k$$

as rational functions (in fact, units over $\tilde{U}_j \cap \tilde{U}_k$). But the ratios Z_j/Z_k are determined by the relations (3.3.7), while G_j/G_k can be computed from (4.2.2), and it is trivial to check that these agree.

Now we can easily complete the proof of Theorem 1. The existence of γ, together with the universal property of blowing up, yields a morphism

$$Bc_m : OH_m \to B_{OD^m} X^m_B$$

which is clearly proper and birational, hence surjective. On the other hand, the fact that the G's generate the ideal of OD^m, and correspond to the Z coordinates on $OH_m \subset X^m_B \times \mathbb{P}^{m-1}$, implies that Bc_m is a closed immersion. Therefore Bc_m is an isomorphism. □

4.4. **Complements and consequences.**

Corollary 4.3. *The image of the relative symmetric product $X^{(m)}_B$ under the elementary symmetric functions embedding σ (cf. Lemma 3.1) is schematically defined by the equations (3.1.1-3.1.2).*

Proof. We have a diagram locally

$$\begin{array}{ccc} H^m & \subset & \mathbb{P}^{m-1} \times \mathbb{A}^{2m} \times B \\ \downarrow & & \downarrow \\ X^{(m)}_B & \stackrel{\sigma}{\hookrightarrow} & \mathbb{A}^{2m} \times B. \end{array} \tag{4.4.1}$$

We have seen that the image of the top inclusion is defined by the equations (3.2.4), (3.3.7). The equations of the schematic image of σ are obtained by eliminating the Z coordinates from the latter equations, and this clearly yields the equations as claimed. □

Now as one byproduct of the proof of Theorem 1.1, we obtained generators of the ideal of the ordered half-discriminant OD^m. As a further consequence, we can determine the ideal of the discriminant locus D^m in the symmetric product $X^{(m)}_B$ itself: let δ^x_m denote the discriminant of F_0, which, as is well known [4], is a polynomial in the σ^x_i such that

$$\delta^x_m = G_1^2. \tag{4.4.2}$$

Set

$$\eta_{i,j} = \frac{(\sigma^y_m)^{i+j-2}}{t^{(i-1)(m-i)+(j-1)(m-j)}} \delta^m_x. \tag{4.4.3}$$

It is easy to see that this is a polynomial in the $\sigma^x_.$ and the $\sigma^y_.$, such that $\eta_{i,j} = G_i G_j$.

Corollary 4.4. *The ideal of D^m is generated, locally near mp, by $\eta_{i,j}, i, j = 1, ..., m$.*

Proof. This follows from the fact that ϖ_m is flat and that

$$\varpi_m^*(\eta_{i,j}) = G_i G_j, i, j = 1, ..., m$$

generate the ideal of $2OD^m = \varpi_m^*(D^m)$. □

No we are in position to make good Remark 2.1:

Lemma 4.5. *$c_m^{-1}(D^m)$ has no embedded points.*

Proof. Consider the function $\eta_{i,i}$, a priori a rational function on $X_B^{(m)}$. Because it pulls back to the regular function G_i^2 on $X_B^{\lceil m \rceil}$, it follows that $\eta_{i,i}$ is in fact regular near the 'origin' mp. Clearly $\eta_{i,i}$ vanishes on the discriminant D^m. Now the divisor of $\eta_{i,i}$ on U_i pulls back via ϖ_m to the divisor of G_i^2, which also coincides with $\varpi_m^*(\Gamma^{(m)})$. Because ϖ is finite flat, it follows the the divisor of $\eta_{i,i}$ coincides over U_i with $\Gamma^{(m)}$ and in particular, the pullback of the ideal of D^m has no embedded component in U_i. Since the $U_i, i = 1, ..., m$ cover a neighborhood of the exceptional locus in $X_B^{[m]}$, this shows $c_m^{-1}(D^m)$ has no embedded components, i.e. is Cartier, as claimed. The reader can check that the foregoing proof is logically independent of any results that depend on the statement of Lemma 4.5, so there is no vicious circle. □

Note that the ideal of the Cartier divisor $c_m^*(D^m)$ on $X_B^{[m]}$, that is, $\mathcal{O}_{X_B^{[m]}}(-c_m^*(D^m))$, is isomorphic in terms of our local model $\tilde{H}$ to $\mathcal{O}(2)$ (i.e. the pullback of $\mathcal{O}(2)$ from $\mathbb{P}^{m-1}$). This suggests that $\mathcal{O}(-c_m^*(D^m))$ is divisible by 2 as line bundle on $X_B^{[m]}$, as the following result indeed shows. First some notation. For a prime divisor A on X, denote by $[m]_*(A)$ the prime divisor on $X_B^{[m]}$ consisting of schemes whose support meets A. This operation is easily seen to be additive, hence can be extended to arbitrary, not necessarily effective, divisors and thence to line bundles.

Corollary 4.6. *Set*

$$\mathcal{O}_{X_B^{[m]}}(1) = \omega_{X_B^{[m]}} \otimes [m]_*(\omega_X^{-1}). \tag{4.4.4}$$

Then

$$\mathcal{O}_{X_B^{[m]}}(-c_m^*(D^m)) \simeq \mathcal{O}_{X_B^{[m]}}(2) \tag{4.4.5}$$

and

$$\mathcal{O}_{X_B^{\lceil m \rceil}}(-oc_m^*(OD^m)) \simeq \varpi_m^* \mathcal{O}_{X_B^{[m]}}(1). \tag{4.4.6}$$

Proof. The Riemann-Hurwitz formula shows that the isomorphism (4.4.5) is valid on the open subset of $X_B^{[m]}$ consisting of schemes disjoint from the locus of fibre nodes of π. Since this open is big (has complement of codimension > 1), the iso holds on all of $X_B^{[m]}$. A similar argument establishes 4.4.6.. $\square$

In practice, it is convenient to view (4.4.4) as a formula for $\omega_{X_B^{[m]}}$, with the understanding that $\mathcal{O}_{X_B^{[m]}}(1)$ coincides in our local model with the $\mathcal{O}(1)$ from the $\mathbb{P}^{m-1}$ factor, and that it pulls back over $X_B^{\lceil m \rceil} = X_B^{[m]} \times_{X_B^{(m)}} X_B^m$ to the $\mathcal{O}(1)$ associated to the blow up of the 'half discriminant' OD^m. We will also use the notation

$$\mathcal{O}(\Gamma^{(m)}) = \mathcal{O}_{X_B^{[m]}}(-1), \Gamma^{\lceil m \rceil} = \varpi_m^*(\Gamma^{(m)})$$

with the understanding that $\Gamma^{(m)}$ is Cartier, not necessarily effective, but $2\Gamma^{(m)}$ and $\Gamma^{\lceil m \rceil}$ are effective. Indeed $\Gamma^{(m)}$ is essentially never effective. Nonetheless, $-\Gamma^{(m)}$ is relatively ample on the Hilbert scheme $X_B^{[m]}$ over the symmetric product $X_B^{(m)}$, and will be referred to as the *discriminant polarization.*

5. Study of H_m

We continue our study of the cycle map over a neighborhood of a maximally singular cycle mp with p a fibre node, using the model H_m. Now having determined the structure of c_m along its 'most special' fibre $c_m^{-1}(m(0,0))$, our purpose in this section is to determine its structure along nearby fibres and their variation. Thus we will assume for the rest of this section, unless otherwise stated, that we are in the local situation where B is a smooth curve, with local coordinate t, and the family U/B is the standard degenration $xy = t$.

5.1. **Nearby fibres.** Let U', U'' denote the x, y axes, respectively in $U_0 = X_0 \cap U$, with their respective origins θ', θ''. If the special fibre X_0 is reducible, then U', U'' globalize to (i.e. are open substs of) the two components of the normalization. If X_0 is irreducible, then both U' and U'' globalize to the normalization. For any pair of natural numbers $(a, b), 0 < a + b < m$, set

$$U^{(a,b)} = U'^{(a)} \times U''^{(b)}$$

(which globalizes to a component –the unique one, if X_0 is irreducible– of the normalization of X_0^{a+b}). Then we have a natural map

$$U^{(a,b)} \to \mathrm{Sym}^m(U_0) \subset \mathrm{Sym}^m(U/B)$$

given by

$$(\sum m_i x_i, \sum n_j y_j) \mapsto \sum m_i(x_i, 0) + \sum n_j(0, y_j) + (m - a - b)(0,0).$$

This map is clearly birational to its image, which we denote by $\bar{U}^{(a,b)}$. Thus $U^{(a,b)}$ coincides with the normailzation of $\bar{U}^{(a,b)}$. It is clear that $\bar{U}^{(a,b)}$ is defined by the equations

$$\sigma_m^x = ... = \sigma_{a+1}^x = 0, \sigma_m^y = ... = \sigma_{b+1}^y = 0.$$

A point

$$c \in \bar{U}^{(a,b)} - (\bar{U}^{(a+1,b)} \cup \bar{U}^{(a,b+1)}),$$

i.e. a cycle in which $(0,0)$ appears with multiplicity exactly $n = m - a - b$, is said to be of *type* (a,b). Type yields a natural stratification of the symmetric product $U_0^{(m)}$. Now let $H^{(a,b)}$ be the closure of the locus of schemes whose cycle if of type (a,b). i.e.

$$H^{(a,b)} = \text{closure}(c_m^{-1}(\bar{U}^{(a,b)} - (\bar{U}^{(a+1,b)} \cup \bar{U}^{(a,b+1)}))) \subset H_m \tag{5.1.1}$$

Clearly the restriction of c_m on $H^{(a,b)}$ factors through a map

$$\tilde{c}_m : H^{(a,b)} \to U^{(a,b)},$$
$$\tilde{c}_m = ((\sigma_1^x, ..., \sigma_a^x), (\sigma_1^y, ..., \sigma_b^y))$$

Approaching the 'origin cycle' $m(0,0)$ through cycles of type (a,b), i.e. approaching the point $(a\theta', b\theta'')$ on $U^{(a,b)}$, means that a (resp. b) points are approaching the origin θ' (resp. θ'') along the x (resp. y)-axis. For a cycle c of type (a,b), we have, for all $j \leq b$, that $\sigma_j^y \neq 0, \sigma_{m-j}^x = 0$, hence by the equations (3.3.1) (setting each $a_i = \sigma_{m-i}^x, d_i = \sigma_{m-i}^y$), we conclude $v_j = 0$; thus

$$v_1 = ... = v_b = 0; \tag{5.1.2}$$

similarly, for all $j \leq a$, we have $\sigma_{m-j}^y = 0, \sigma_j^x \neq 0$, hence again by the equations (3.3.1) , we conclude $u_{m-j} = 0$; thus

$$u_{m-1} = ... = u_{m-a} = 0. \tag{5.1.3}$$

Consequently, the fibre of c_m over this point is schematically

$$c_m^{-1}(c) = \tilde{c}_m^{-1}(c) = \bigcup_{i=b+1}^{m-a-1} C_i^m, \tag{5.1.4}$$

provided $a + b \leq m - 2$. If $a + b = m - 1$, the fibre is the unique point given by

$$v_1 = ... = v_b = u_{b+1} = ... = u_{m-1} = 0$$

(as subscheme of X/B, this point is the one denoted Q_{b+1}^m in [7], and has ideal (x^{m-b}, y^{b+1})). As c approaches the 'origin' $(a\theta', b\theta'')$ in $U^{(a,b)}$, the equations (5.1.2),(5.1.3) persist, so we conclude

$$\tilde{c}_m^{-1}((a\theta', b\theta'')) = \begin{cases} \bigcup\limits_{i=b+1}^{m-a-1} C_i^m, a + b \leq m - 2, \\ Q_{b+1}^m, a + b = m - 1. \end{cases} \tag{5.1.5}$$

Thus, working in $H^{(a,b)}$ over $U^{(a,b)}$, the special fibre is the same as the general fibre, so $H^{(a,b)}/U^{(a,b)}$ is a (locally trivial) chain-of-$\mathbb{P}^1$'s -bundle. Moreover, the individual $\mathbb{P}^1$'s are well defined over $U^{(a,b)}$ (by vanishing of suitable u, v coordinates), so $H^{(a,b)}$ splits as a union of $\mathbb{P}^1$-bundles arranged as a chain, where each ith bundle meets the next in a section (corresponding to Q^m_i). These $\mathbb{P}^1$-bundles are the *node scrolls*, to be discussed further below. And in the special case $a + b = m - 2$, we see that $H^{(a,b)}$ forms a single node scroll, with fibre C^m_{b+1}. Also, as subscheme of $H_m \times_{U^{(m)}} U^{(a,b)}$, $H^{(a,b)}$ is defined by the equations (5.1.3) and (5.1.2). Summarizing,

Lemma 5.1. *For any $a + b \le m - 2$, $H^{(a,b)}/U^{(a,b)}$ is a union of $(m - 1 - a - b)$ $\mathbb{P}^1$ bundles arranged in a chain where each meets the next in a section. If $a + b = m - 1$, $H^{(a,b)} \to U^{(a,b)}$ is an isomprphism.*

Note that the case $a + b = m$, i.e. where $U^{(a,b)}$ is an entire component of the boundary of $U^{(m)}_B$, behaves differently. There a similar but simpler analysis shows that the fibre $\tilde{c}_m^{-1}((a\theta', b\theta''))$ coincides with $C^m_b \simeq \mathbb{P}^1$ if $1 \le b \le m - 1$ and with the single point Q^m_{b+1} if $b = 0, m$. This, of course, is contained in part (vi) of Theorem 3.3 above. So in this case $H^{(a,b)}/U^{(a,b)}$ is a blowup and not not a fibre bundle, except in the extreme cases $b = 0, m$, where it is bijective.

Reworking in terms of Z coordinates, we get

Corollary 5.2. *For any $\ell_1 + \ell_2 \le m, \ell_1, \ell_2 \ge 0$, we have, in any component of the locus $c_m^{-1}((U')^{(\ell_1)} \times (U'')^{(\ell_2)}) \subset H_m$ dominating $(U')^{(\ell_1)} \times (U'')^{(\ell_2)}$, that*

$$Z_i = 0, \forall i \ge \max(m - \ell_1 + 1, \ell_2 + 2), \quad \forall i \le \min(\ell_2, m - \ell_1 - 1). \tag{5.1.6}$$

In particular, if $\ell_1 + \ell_2 = m$ (resp. $\ell_1 + \ell_2 < m$), the only nonzero Z_i are where

$$i \in [\ell_2, \ell_2 + 1] \cap [1, m] \ \ (\textit{resp. } \ i \in [l_2 + 1, m - \ell_1] \cap [1, m]).$$

Taking to account the linear relations (3.3.7), we also conclude

Corollary 5.3. (i) *For any component $(U')^\ell \times (U'')^{m-\ell}, 0 < \ell < m$ of the special fibre of $U^{(m)}_B$, the unique dominant component of $c_m^{-1}((U')^\ell \times (U'')^{m-\ell})$ coincides with the graph of rational map*

$$(U')^\ell \times (U'')^{m-\ell} \dashrightarrow \mathbb{P}^1_{Z_\ell, Z_{\ell+1}} \subset \mathbb{P}^{m-1}_Z \tag{5.1.7}$$

defined by

$$[\sigma^x_\ell, \sigma^y_{m-\ell}];$$

(ii) *ditto for $\ell = m$ (resp. $\ell = 0$), with the constant rational map to $[1, 0, ...]$ (resp. $[..., 0, 1]$);*
(iii) *ditto over $(U')^\ell \times (U'')^{m-\ell-1}, 0 \le \ell \le m - 1$, with the constant map to $[..., 0, 1_{m-\ell}, 0...]$.*

5.2. **Comparing H_m with H_n.** Now on the other hand, working near a cycle c of type (a,b) and fixing its off-node portion, say of length $k = m - n$, we also have an obvious identification of the same (general) fibre of $H^{(a,b)}/U^{(a,b)}$ over c as the special fibre in a local model H_n for the length-n Hilbert scheme. Namely, if we let $c' = n(0,0)$ be the part of c supported at the origin, then essentially the same fibre $c_m^{-1}(c)$ can also be written as

$$c_n^{-1}(c') = \bigcup_{j=1}^{n} C_j^n \tag{5.2.1}$$

and naturally C_j^n corresponds to $C_{j+b}^m = C_{j+b}^{n+a+b}$. Of course under the identification of Theorem 3.3, $c_n^{-1}(c')$ corresponds to the punctual Hilbert scheme $Hilb_n^0(X_0)$. So we conclude that the j-th punctual length-n Hilbert scheme component at c specializes to the $(j+b)$-th length-m Hilbert scheme component at $m(0,0)$ as c specializes to $m(0,0)$ over the normalization $X^{(a,b)}$. Note that the analogous fact holds for any cycle of multiplicity n at $(0,0)$ specializing to one of multiplicity m at $(0,0)$, even if its total degree is higher. Thus we can conclude

Lemma 5.4. *As a cycle c on X_0, having multiplicity n at the origin, approaches a cycle d with multiplicity $m > n$ at the origin, so that for some a, b with $a + b = m - n$, a points approach along the x-axis and b points along the y-axis, the punctual Hilbert scheme component C_j^n over c specializes smoothly to C_{j+b}^m at d.*

(The statement being local, the general case follows from that of the local model). We also see, comparing (5.1.4) and (5.1.5), that the fibre $\tilde{c}_m^{-1}(c)$ is 'constant', i.e. it doesn't depend on c as it moves in $U^{(a,b)}$. Moreover, as c moves in $U^{(a,b)}$, the individual components of this fibre, which have to do with branches of $U_0^{(m-a-b)}$ at $(m-a-b)(0,0)$, U_0 being the entire singular fibre (or what is the same, branches of U_0^m generically along $U^{(a,b)}$), remain well defined (i.e. not interchanged by monodromy), and specialize smoothly to similar components on lower-dimensional strata. Note that this is true even if U_0 is replaced by X_0 which is (globally) irreducible, in which case the other $a+b$ branches of X_0, a from U' and b from U'' are globally interchangeable. Therefore we can refine slightly the statement of Lemma 5.1:

Lemma 5.5. *for any $a + b \leq m - 2$, we have*

$$H^{(a,b)} = \bigcup_{j=1}^{n-1} F_j^{(a|b)} \tag{5.2.2}$$

where $F_j^{(a,b)} \subset H_m$ is the subscheme defined by

$$v_1 = ... = v_{j+b} = u_{j+b+1} = ... = u_{m-1} = 0 \tag{5.2.3}$$

and

$$\tilde{c}_m : F_j^{(a|b)} \to U^{(a,b)}$$

is a $\mathbb{P}^1$ *bundle with fibre* C_j^n*. Moreover,* $\tilde{c}_m|_{F^{(j:a|b)}}$ *admits two disjoint sections* Q_j^n, Q_{j+1}^n *with respective fibres the points corresponding to the punctual schemes of the same type, and* $F^{(j:a|b)} \cap F^{(j+1:a|b)} = Q_{j+1}^n$.

5.3. **Node scrolls: a preview.** Fixing a, b for now, the $F_j = F_j^{(a|b)}$ are special (but typical) cases of what are called *node scrolls*. It follows from the last lemma that we can write

$$F_j = \mathbb{P}(L_j^n \oplus L_{j+1}^n)$$

for certain line bundles L_j^n on $U^{(a,b)}$, corresponding to the disjoint sections Q_j^n, Q_{j+1}^n, where the difference $L_j^n - L_{j+1}^n$ is uniquely determined (we use additive notation for the tensor product of line bundles and quotient convention for projective bundles). The identification of a natural choice for both these line bundles, using methods to be developed later in this section, will be taken up in the next section and plays an important role in the enumerative geometry of the Hilbert scheme. But the difference $L_j^n - L_{j+1}^n$, and hence the intrinsic structure of the node scroll F_j, may already be computed now, as follows.

Write

$$Q_j = \mathbb{P}(L_j), Q_{j+1} = \mathbb{P}(L_{j+1})$$

for the two special sections of type Q_j^n, Q_{j+1}^n respectively. Let

$$D_{\theta'}, D_{\theta''} \subset U^{(a,b)}$$

be the divisors comprised of cycles containing θ' (resp. θ''). In the local model, these are given locally by the respective equations

$$D_{\theta'} = (\sigma_a^x), D_{\theta''} = (\sigma_b^y).$$

Lemma 5.6. *We have, using the quotient convention for projective bundles,*

$$F_j = \mathbb{P}_{U^{(a,b)}}(\mathcal{O}(-D_{\theta'}) \oplus \mathcal{O}(-D_{\theta''})), j = 1, ..., n-1. \tag{5.3.1}$$

Proof. Our key tool is a $\mathbb{C}^*$- parametrized family of sections 'interpolating' between Q_j and Q_{j+1}. Namely, note that for any $s \in \mathbb{C}^*$, there is a well-defined section I_s of F_j whose fibre over a general point $z \in X^{(a,b)}$ is the scheme

$I_s(z) = (sx^{n-j} + y^j)$ [redacted] $\mathrm{sch}(z)$,

where $\mathrm{sch}(z)$ is the unique subscheme of length $a+b$, disjoint from the nodes, corresponding to z.

Claim: The fibre of I_s over a point $z \in D_{\theta'}$ (resp. $z \in D_{\theta''}$) is Q_j^n (resp. Q_{j+1}^n).

Proof of claim. Indeed set-theoretically the claim is clear from the fact the this fibre corresponds to a length-n punctual scheme meeting the x-axis (resp. y-axis) with multiplicity at least $n-j+1$ (resp. $j+1$).

To see the same thing schematically, via equations in the local model H_{n+1}, we proceed as follows. Working near a generic point $z_0 \in D_{\theta'}$ we can, discarding distal factors supported away from the nodes, write the singleton scheme corresponding nearby cycle z as $\mathrm{sch}(z) = (x - c, y)$ where $c \to 0$ as $z \to z_0$, and then

$$I_s(z) = (sx^{n-j} + y^j)(x - c, y) = (sx^{n-j+1} - csx^{n-j} - cy^j, y^{j+1}).$$

Thus, in terms of the system of generators (3.3.2), $I_s(z)$ is defined locally by

$$cu_j - sv_j = 0 \tag{5.3.2}$$

(with other $[u_k, v_k]$ coordinates either $[1, 0]$ for $k < j$ or $[0, 1]$ for $k > j$. The limit of this as $c \to 0$ is $[u_j, v_j] = [1, 0]$, which is the point Q_j. *QED Claim.*

Clearly I_s doesn't meet Q_j or Q_{j+1} away from $D_{\theta'} \cup D_{\theta''}$. Therefore, we have

$$I_s \cap Q_j = Q_j.D_{\theta'}, \tag{5.3.3}$$

$$I_s \cap Q_{j+1} = Q_{j+1}.D_{\theta''}; \tag{5.3.4}$$

an easy calculation in the local model shows that the intersection is transverse. Because $Q_j \cap Q_{j+1} = \emptyset$, it follows that

$$I_a \sim Q_j + D_{\theta'}.F_j \tag{5.3.5}$$
$$\sim Q_{j+1} + D_{\theta''}.F_j. \tag{5.3.6}$$

These relations also follow from the fact, which comes simply from setting $s = 0$ or dividing by s and setting $s = \infty$ in (5.3.2), that

$$\lim_{s \to 0} I_s = Q_j + D_{\theta'}.F_j, \lim_{s \to \infty} I_s = Q_{j+1} + D_{\theta''}.F_j \tag{5.3.7}$$

It then follows that

$$(Q_j)^2 = Q_j.(I_s - D_{\theta'}.F_j) = Q_j.(Q_{j+1} + (D_{\theta''} - D_{\theta'}).F_j),$$

hence

$$(Q_j)^2 = Q_j(D_{\theta''} - D_{\theta'}), \tag{5.3.8}$$

therefore finally

$$L_j^n - L_{j+1}^n = D_{\theta''} - D_{\theta'}. \tag{5.3.9}$$

This proves the Lemma. □

6. Definition of node scrolls and polyscrolls

We now begin to extend our scope to a global proper family X/B of nodal curves, with possibly higher-dimensional base and fibres with more than one node. Our main interest is in the node scrolls in this generality, where, rather than living over a symmetric product, they become $\mathbb{P}^1$-bundles over a relative Hilbert scheme (of lower degree) associated to a 'boundary family' of X/B, i.e a family obtained, essentially, as the partial normalization of the subfamily of X/B lying over the normalization of a component of the locus of singular curves in B (a.k.a. the boundary of B). For our purposes, it will be convenient to work 'node by node', associating to each a boundary family. We begin by making the appropriate notion of boundary family precise.

6.1. **Boundary data.** Let $\pi : X \to B$ now denote an arbitrary flat family of nodal curves of arithmetic genus g over an irreducible base, with smooth generic fibre. In order to specify the additional information required to define a node scroll, we make the following definition.

Definition 6.1. *A boundary datum for X/B consists of*

(i) *an irreducible variety T with a map $\delta : T \to B$ unramified to its image;*
(ii) *a 'relative node' over T, i.e. a lifting $\theta : T \to X$ of δ such that each $\theta(t)$ is a node of $X_{\delta(t)}$;*
(iii) *a labelling, continuous in t, of the two branches of $X_{\delta(t)}$ along $\theta(t)$ as x-axis and y-axis.*

Given such a datum, the associated boundary family X_T^θ is the normalization (= blowup) of the base-changed family $X \times_B T$ along the section θ, i.e.

$$X_T^\theta = \mathrm{B}\ell_\theta(X \times_B T),$$

viewed as a family of curves of genus $g-1$ with two, everywhere distinct, individually defined marked points θ_x, θ_y. We denote by ϕ the natural map fitting in the diagram

$$\begin{tikzcd} X_T^\theta \arrow[d] \arrow[dr, "\phi"] & \\ X \times_B T \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r, "\delta"] & B. \end{tikzcd}$$

Note that a boundary datum indeed lives over the boundary of B; in the other direction, we can associate to any component T_0 of the boundary of B a finite number boundary data in this sense: first consider a component T_1 of the normalization of $T_0 \times_B \mathrm{sing}(X/B)$, which already admits a node-valued lifting θ_1 to X, then further base-change by the normal cone of $\theta_1(T_1)$ in X (which is 2:1 unramified, possibly disconnected, over T_1), to obtain a boundary datum as above. 'Typically', the curve corresponding to a general point in T_0 will have a single node θ and then the degree of δ will be 1 or 2 depending on whether the branches along θ are distinguishable in X or not (they always are distinguishable if θ

is a separating node and the separated subcurves have different genera). Proceeding in this way and taking all components which arise, we obtain finitely many boundary data which 'cover', in an obvious sense, the entire boundary of B. Such a collection, weighted so that each boundary component T_0 has total weight $= 1$ is called a *covering system of boundary data*.

6.2. **Node scrolls: definition.**

Proposition-definition 6.2. *Given a boundary datum (T, δ, θ) for X/B and natural numbers $1 \leq j < n$, there exists a $\mathbb{P}^1$-bundle $F_j^n(\theta)$, called a* <u>*node scroll*</u> *over the Hilbert scheme $(X_T^\theta)^{[m-n]}$, endowed with two disjoint sections $Q_{j,j}^n, Q_{j+1,j}^n$, together with a surjective map generically of degree equal to* $\deg(\delta)$ *of*

$$\bigcup_{j=1}^{n-1} F_j^n(\theta) := \underset{j=1}{\overset{n-1}{\blacksquare}} F_j^n(\theta) / \underset{j=1}{\overset{n-2}{\blacksquare}} (Q_{j+1,j}^n \sim Q_{j+1,j+1}^n)$$

onto the closure in $X_B^{[m]}$ of the locus of schemes having length precisely n at θ, so that a general fibre of $F_j^n(\theta)$ corresponds to the family C_j^n of length-n schemes at θ generically of type $I_j^n(a)$, with the two nonprincipal schemes Q_j^n, Q_{j+1}^n corresponding to $Q_{j,j}^n, Q_{j,j+1}^n$ respectively. We denote by δ_j^n the natural map of $F_j^n(\theta)$ to $X_B^{[m]}$.

Proof-construction. The scroll $F_j^n(\theta)$ is defined as follows. Fixing the boundary data, consider first the locus

$$\bar{F}_j^n \subset T \times_B X_B^{[m]}$$

consisting of compatible pairs (t, z) such that z is in the closure of the set of schemes which are of type I_j^n (i.e. $x^{n-j} + ay^j, a \in \mathbb{C}^*$) at $\theta(t)$, with respect to the branch order (θ_x, θ_y). The discussion of the previous subsection shows that the general fibre of $\bar{F}_j$ under the cycle map is a $\mathbb{P}^1$, namely a copy of C_j^n; moreover the closure of the locus of schemes having multiplicity n at θ is the union $\bigcup\limits_{j-1}^{n-1} \bar{F}_j^n$. In fact locally over a neighborhood of a cycle having multiplicity precisely $n + e$ at θ, $\bar{F}_j^n$ is a union of components $\bar{F}_j^{(n:a,b)} \times U^{(m-e)}, a + b = e$, where U is an open set disjoint from θ, $\bar{F}_j^{(n:a,b)} \subset H_{n+e}$ maps to $(X')^a \times (X'')^b$ and is defined in H_{n+e} by by the vanishing of all $Z_i, i \neq j + b, j + b + 1$ or alternatively, in terms of u, v coordinates, by

$$v_1 = ... = v_{j+b} = u_{j+b+1} = ... = u_{n+e} = 0$$

Then $F_j^n(\theta)$ is the locus

$$\{(w, t, z) \in (X_T^\theta)^{[m-n]} \times_T \bar{F}_j^n : \phi_*(c_{m-n}(w)) + n\theta = c_m(z)\}, \tag{6.2.1}$$

where $\phi : X^\theta \to X$ is the natural map, clutching together θ_x and θ_y, and ϕ_* is the induced push-forward map on cycles. Then the results of the previous section show that $F^n_j(\theta)$ is locally defined near a cycle having multiplicity b at θ_y, e.g. by the vanishing of the $Z_i, i \neq j+b, j+b+1$ on

$$\{(w, u, Z) \in (X^\theta_T)^{[m-n]} \times X^{(e)}_B \times \mathbb{P}^{n+e} : \phi_*(c_{m-n}(w))_\theta + n\theta = u\}$$

where $\_\theta$ indicates the portion near θ. The latter locus certainly projects isomorphically to its image in $(X^\theta_T)^{[m-n]} \times \mathbb{P}^{n+e}$, hence $F^n_j(\theta$ is a $\mathbb{P}^1$-bundle over $(X^\theta_T)^{[m-n]}$. Since $F^n_j(\theta)$ admits the two sections $Q^n_{j,j}, Q^n_{j+1,j}$, it is the projectivization of a decomposable rank-2 vector bundle. □

Note that the node scroll $F^n_j(\theta)$ also depends on m, and is by construction a subscheme of the 'flag' Hilbert scheme

$$\begin{array}{cccc} F^n_j(\theta) \subset & \{(z_1, z_2) : \phi(z_1) \subset z_2\} & \to X^{[m]}_B \\ & \downarrow & \\ & (X^\theta_T)^{[m-n]} & \end{array} \tag{6.2.2}$$

We will denote the two Hilbert-scheme targeted projections on F^n_j by $p_{[m-n]}, p_{[m]}$ respectively. When the dependence on m needs to be made explicit, we will write $F^{n,m}_j$. The following simple technical point will be needed below.

Lemma 6.3. *Let $T' \to T$ be a base change and θ' a smooth section of $X^\theta_{T'}$ disjoint from $\theta_{T'}$ and identified with a section of $X^\theta_{T'}$. Then on the pulled- back node scroll $F^n_j(\theta)_{T'}$,*

$$p^*_{[m]}[m]_*\theta' = p^*_{[m-n]}[m-n]_*\theta'$$

Proof. It suffices to verify this on the ordered version where, e.g. $[m]_*\theta' = \sum\limits_{i=1}^{m} p^*_i\theta'$. But for $i = 1, ..., n$, the indices corresponding to the fibre of the node scroll F^n, we have $p^*_i\theta' \cap F = \emptyset$ as the nodes are disjoint. This gives our assertion. □

Obviously, $Q^n_{j,j-1}$ and $Q^n_{j,j}$ coincide in $(X^\theta_T)^{[m-n]} \times X^{[m]}_B$ and when convenient we will write them as Q^n_j or $Q^{n,m}_j$. It is noteworthy that the map from Q^n_j can be written down explicitly:

Lemma 6.4. *The map $(X^\theta_T)^{[k]} \simeq Q^n_j \to X^{[m]}_B$ is given by*

$$z_0 + a_x\theta_x + a_y\theta_y \mapsto \phi(z_0) + Q^{n+a_x+a_y}_{j+a_y} \tag{6.2.3}$$

where z_0 is supported off $\theta_x \cup \theta_y$.

Proof. To begin with, as θ_x, θ_y are smooth sections of X_T^θ, any length-k subscheme of it can indeed be expressed uniquely as in the formula. The formula is clearly true When $a_x = a_y = 0$. Then the general case follows by taking limits, in view of the explicit local description of the schemes of type Q_r^p as (x^{p-r+1}, y^r). □

In addition to the node scroll $F_j^n(\theta)$, we will also consider its ordered version, i.e.

$$OF_j^n(\theta) = F_j^n(\theta) \times_{(X_T^\theta)^{(m-n)}} (X_T^\theta)^{m-n}, \tag{6.2.4}$$

and and similarly for $\bar{OF}_j^n(\theta)$. Also, for each n-tuple $I \subset [1, m]$, the corresponding locus in $X_B^{\lceil m \rceil}$, i.e.

$$OF_j^I = \{(w, t, z) \in (X_T^\theta)^{\lceil m-n \rceil} \times_T \bar{OF}_j^n : \phi_*(oc_{m-n}(w)) + \sum_{i \in I} p_i^*(\theta) = c_m(z)\}, \tag{6.2.5}$$

this being the 'node scroll inserted over the I-indexed coordinates.

6.3. **Polyscrolls.** Consider now a collection $\theta_. = (\theta_1, ..., \theta_r)$ of distinct relative nodes of X/B and $T = T(\theta_1, ..., \theta_r) \to B$ a common boundary locus for them, compatible with the boundary data for each θ_i. Let $X_T^{\theta.}$ be the blowup or partial normalization of X_T in $\theta_1, ...\theta_r$. As the θ_i are disjoint, the blowing up may be done inductively, in any order, or simultaneously. Let $(j.), (n.)$ be sequences of r positive integers with $(j.) < (n.)$ in the sense that $j_i < n_i, \forall i$. We aim to define a node polyscroll $F = F_{j.}^{n.,m}(\theta.; X/B)$. This can be done using induction on r. Assume the $(r-1)$- polyscroll $F' = F_{j_2,...,j_r}^{n_2,...,n_r,m-n_1}(\theta_2, ..., \theta_r; X_{T(\theta_1)}^{\theta_1})$ is defined, together with maps

$$\begin{array}{ccc} & F' & \xrightarrow{p_{[m-n_1]}} (X^{\theta_1})_{T(\theta_1)}^{[m-n_1]} \\ p_{[m-|n.|]} & \downarrow & \\ & (X_T^{\theta.})^{[m-|n.|]} & \end{array}$$

the horizontal one one being generically finite and the vertical one a $(\mathbb{P}^1)^{r-1}$-bundle projection. Of course, the node scroll $F_{j_1}^{n_1,m}(\theta_1; X/B)$ is a $\mathbb{P}^1$-bundle over $(X^{\theta_1})_{T(\theta_1)}^{[m-n_1]}$. Define F as the fibre product

$$\begin{tikzcd} & F \arrow[dl] \arrow[dr] & & \\ F_{j_1}^{n_1,m}(\theta_1; X/B) \arrow[dr] & \diamondsuit & F' \arrow[dl] \arrow[d] \\ & (X^{\theta_1})_{T(\theta_1)}^{[m-n_1]} & (X_T^{\theta.})^{[m-|n.|]} \end{tikzcd} \tag{6.3.1}$$

Then F comes equipped with a $(\mathbb{P}^1)^r$-bundle projection $p_{[m-|n.|]}F \to F' \to (X_T^{\theta.})^{[m-|n.|]}$, as well as a generically finite map $p_{[m-n_1]} : F \to F_{j_1}^{n_1,m}(\theta_1; X/B) \to X_B^{[m]}$. Writing, suggestively, F' as $F(\hat{\theta}_1)$ and assuming inductively maps $F' \to F'(\hat{\theta}_i), \forall i > 1$, we can identify

$F^{n_1,m}_{j_1}(\theta_1; X/B) \times F'(\hat{\theta}_i)$ as $F(\hat{\theta}_i)$ and obtain an induced map $F \to F(\hat{\theta}_i)$. Then taking fibre product with $F^{n_i,m}_{j_i}(\theta_i; X/B)$, we obtain a morphism, easily seen to be an isomorphism, from F to a similar node polyscroll with θ_1, θ_i interchanged. Continuing in this way, it is easy to see that F is independent of the ordering and the composite $F \to F' \to (X^{\theta.}_T)^{[m-|n.|]}$ is a $(\mathbb{P}^1)^r$-bundle.

Ordered variants can be constructed similarly.

We summarize some of the important properties of node polyscrolls as follows

Proposition 6.5. (i) *The r-polyscroll $F = F^{n.,m}_{j.}(\theta..X/B)$ is a $(\mathbb{P}^1)^r$-bundle over the Hilbert scheme $(X^{\theta.}_T)^{[m-|n.|]}$.*

(ii) *F parametrizes subschemes of X/B having length at least n_i at $\theta_i, i = 1, ..., r$.*

(iii) *F is independent of the order of $(\theta., n., j.)$ and admits a $(\mathbb{P}^1)^{r-s}$-bundle projection to a pullback of the s-polyscroll based on any s of the (θ_i, n_i, j_i).*

7. Structure of node scrolls

We aim to determine the global structure of the node scrolls. We will do this first for the ordered version of the Hilbert scheme, viz $X^{\lceil m\rceil}_B$ with its ordered cycle map to X^m_B. To this end, a key point is the globalization of the G functions on X^m_B, or rather their divisors of zeros, over the cycles supported in a neighborhood of a relative node of X/B, then over the entire symmetric product. This generalization will take the form of certain decomposable vector bundles called G-bundles.

7.1. **G- bundles.** The G-bundles are direct sums of line bundles corresponding to a chain of m essentially canonical 'intermediate diagonal' divisors, interpolating between the 'x-discriminant' and the 'y-discriminant'. These intermediate diagonals are Cartier divisors consisting of the big diagonal OD^m plus certain boundary divisors, and the common schematic intersection of all of them is exactly OD^m. This gives a (decomposable) vector bundle surjecting to the ideal of the discriminant.

Proposition 7.1. *Let X/B be a flat family of nodal curves with irreducible base and generic fibre. Let θ be a relative node of X/B. Then*

(i) *there exists an analytic neighborhood U of θ in X and a rank-m vector bundle $G^m(\theta)$, defined in $U^m_B \subset X^m_B$, together with a surjection in U^m_B:*

$$G^m(\theta) \to \mathcal{I}_{OD^m} \tag{7.1.1}$$

giving rise to a natural polarized embedding

$$U^{\lceil m\rceil}_B = \mathrm{B}\ell_{OD^m}(U^m_B) \hookrightarrow \mathbb{P}(\mathcal{I}_{OD^m}|_{U^m_B}) \to \mathbb{P}(G^m(\theta)). \tag{7.1.2}$$

(ii) *For any relatively affine, étale open $\tilde{U} \to U$ of θ in X/B in which the 2 branches along θ are distinguishable, $G^m(\theta)$ splits over $(\tilde{U})^m$ as a direct sum of invertible ideals $G^m_j(\theta), j = 1, ..., m$;*

(iii) *Moreover if $V = U \setminus \pi^{-1}\pi(\theta)$, i.e. the union of the smooth fibres in U, the restriction of each $G^m_j(\theta)$ on V^m_B is isomorphic to $\mathcal{I}_{OD^m}$.*

Proof-construction. Fix a boundary datum (T, δ, θ) corresponding to θ as in §6. We first work locally in X, near a node in one singular fibre. Then we may assume the two branches along θ are distinguishable in U. We let β_x, β_y denote the x and y branches, locally defined respectively by $y = 0, x = 0$. Consider the Weil divisor on U^m_B defined (manifestly canonically) by

$$OD^m_x(\theta) = OD^m + \sum_{i=2}^{m} \binom{i}{2} \sum_{\substack{I \subset [1,m] \\ |I| = i}} p^*_I(\beta_y) p^*_{I^c}(\beta_x). \tag{7.1.3}$$

Claim 7.2. *We have*

$$OD^m_x(\theta) = \mathrm{zeros}(G_1). \tag{7.1.4}$$

where G_1 denotes the locally-defined Van der Monde determinant with respect to a local coordinate system as above.

Proof of claim. Indeed each factor $x_a - x_b$ of G_1 vanishes on $p^*_I(\beta_y)p^*_{I^c}(\beta_x)$ precisely when $a, b \in I$; the rest is simple counting. □

As $OD^m_x(\theta)$ is canonical, it follows that the divisor of G_1 is canonically defined, depending only on the choice of branch. Given this, it is natural in view of (4.2.1) to define the j-th *intermediate diagonal* along θ as

$$OD^m_{x,j}(\theta) = OD^m_x(\theta) + (j-1)\sum p^*_i(\beta_x) - (j-1)(m - j/2)\partial_\theta \tag{7.1.5}$$

where $\partial_\theta = \beta_x + \beta_y$ is the boundary divisor corresponding to the node θ. Indeed (4.2.1) now shows

$$OD^m_{x,j}(\theta) = \mathrm{zeros}(G_j). \tag{7.1.6}$$

In particular, it is an effective Cartier divisor on U^m_B. Though each individual intermediate diagonal depends on the choice of branch, the collection of them does not. Indeed the elementary identity

$$\sigma^y_m V^m_1 = (-t)^{\binom{m}{2}} V^m_m. \tag{7.1.7}$$

shows that flipping x and y branches takes $-OD^m_{x,j}(\theta)$ to $-OD^m_{y,m+1-j}(\theta)$. Now set

$$G^m(\theta) = \bigoplus_{j=1}^{m} G^m_j(\theta), \text{ where } G^m_j(\theta) = \mathcal{O}(-OD^m_{x,j}(\theta)) \tag{7.1.8}$$

This rank-m vector bundle is independent of the choice of branch, as is the natural map $G^m(\theta) \to \mathcal{O}_{U^m_B}$. Therefore these data are defined globally over B in a suitable analytic

neighborhood of θ^m. By Corollary 4.2, the image of this map is precisely the ideal of OD^m, i.e. there is a surjection

$$G^m(\theta) \to \mathcal{I}_{OD^m}|_{U_B^m} \to 0. \tag{7.1.9}$$

Applying the $\mathbb{P}$ functor, we obtain a closed embedding

$$\mathbb{P}(\mathcal{I}_{OD^m}) \to \mathbb{P}(G^m(\theta)) \tag{7.1.10}$$

Now the blow-up of the Weil divisor OD^m, which we have shown coincides with the Hilbert scheme $U_B^{\lceil m \rceil}$, is naturally a subscheme of $\mathbb{P}(\mathcal{I}_{OD^m})$, whence a natural embedding

$$U_B^{\lceil m \rceil} \to \mathbb{P}(G^m(\theta)) \tag{7.1.11}$$

Note that this is well-defined globally over U_B^m, which sits over a neighborhood of the boundary component in B corresponding to θ, i.e. ∂_θ. □

It is important to record here for future reference a compatibility between the $G_j^m(\theta)$ for different m's. To this end let $U_x, U_y \subset U$ denote the complement of the y (resp. x) branch, i.e. the open sets given by $x \neq 0, y \neq 0$.

Lemma 7.3 (Localization formula)**.** *We have for all $1 \le j \le m - k_x - k_y$,*

$$\begin{gathered} G^m_{j+k_y}(\theta)|_{U^{m-k_x-k_y}\times U_x^{k_x}\times U_y^{k_y}} = \\ p^*_{U^{m-k_x-k_y}} G_j^{m-k_x-k_y}(\theta) \otimes \prod_{a<b>m-k_x-k_y} p^*_{a,b}(d_{a,b}) \end{gathered} \tag{7.1.12}$$

where $d_{a,b}$ is an equation for the Cartier divisor which equals the diagonal in a, b coordinates; in divisor terms, this means

$$OD^m_{x,j+k_y}|_{U^{m-k_x-k_y}\times U_x^{k_x}\times U_y^{k_y}} = p^*_{U^{m-k_x-k_y}}(OD_j^{m-k_x-k_y}) + \sum_{a<b>m-k_x-k_y} p^*_{a,b}(OD^2) \tag{7.1.13}$$

where the last sum is Cartier and independent of j.

Proof. We begin with the observation that, for the universal deformation X_B of a node p, given by $xy = t$, the Cartesian square X_B^2 is nonsingular away from (p, p), hence the diagonal is Cartier away from (p, p), defined locally by $x_1 - x_2$ in the open set where $x_1 \neq 0$ or $x_2 \neq 0$ and likewise for y. Because the question is local and locally any deformation is induced by the universal one, a similar assertion holds for an arbitrary family. Now returning to our situation, let us write $n = m - k_x - k_y$ and N, K_x, K_y for the respective index ranges $[1, n], [n + 1, n + k_x], [n + k_x + k_y + 1, m]$, and x^N, y^N etc. for the corresponding monomials. Then $x_a - x_b$ is a single defining equation for the Cartier (a, b) diagonal whenever a or b is in K_x. Then by (4.2.1), (4.2.2), we can write $G^m_{j+k_y}(\theta)$, up to a unit, i.e. a

function vanishing nowhere in the open set in question, in the form

$$\text{(7.1.14)} \qquad \frac{G_1^N (y^N)^{j-1}}{t^{(j-1)(n-j/2)}} \prod_{\substack{a<b \\ a \text{ or } b \in K_x}} (x_a - x_b) \frac{(y^N)^{k_y} \prod_{a\in N, b\in K_y} (x_a - x_b)}{t^{nk_y}} \frac{\prod_{a<b\in K_y} (x_a - x_b)}{t^{\binom{k_y}{2}}}$$

where we have used the fact that $\frac{(y^{K_x})^{j+k_y-1}}{t^{k_x(j+k_y-1)}}$ is a unit. Now in (7.1.14), the first factor is just G_j^N while the second is the equation of a Cartier partial diagonal. The third factor is equal up to a unit to $\prod_{a\in N, b\in K_y} (y_a - y_b)$, hence is also the equation of a Cartier partial diagonal. Finally, in the fourth factor, each subfactor $(x_a - x_b)/t = y_b^{-1} - y_a^{-1}$, so this too yields a Cartier partial diagonal. □

Now recall the notion of boundary datum (T, δ, θ) introduced in §6. We are now in position to determine globally the pullback of the intermediate diagonals to the partial normalization X_T^θ:

Corollary 7.4. (i) *The pullback of $G^m_{j+k_y}(\theta)$ on $U_B^{m-k_x-k_y} \otimes_B U_B^{\lceil k_x+k_y \rceil}$ extends over $U_B^{m-k_x-k_y} \otimes_B (X_T^\theta)_T^{\lceil k_x+k_y \rceil}$ to*

$$\text{(7.1.15)} \qquad p^*_{U^{m-k_x-k_y}} G_j^{m-k_x-k_y}(\theta) \otimes p^*_{X_B^{\lceil k_x+k_y \rceil}} \mathcal{O}(-\Gamma^{\lceil k_x+k_y \rceil}) \otimes \bigotimes_{a\le m-k_x-k_y<b} p^*(\mathcal{O}(-OD^m_{a,b}))$$

where the last factor is invertible;

(ii) *the closure in $U_B^{m-k_x-k_y} \otimes_B (X_T^\theta)_T^{\lceil k_x+k_y \rceil}$ of the pullback of $OD^m_{j+k_y}(\theta)$ to $U_B^{m-k_x-k_y} \otimes_B U_B^{\lceil k_x+k_y \rceil}$ equals*

$$\text{(7.1.16)} \qquad p^*_{U^{m-k_x-k_y}} OD_j^{m-k_x-k_y}(\theta) + p^*_{X_B^{\lceil k_x+k_y \rceil}} (\Gamma^{\lceil k_x+k_y \rceil}) + \sum_{a\le m-k_x-k_y<b} p^*(OD^m_{a,b})$$

where each summand is Cartier.

Proof. The first assertion is immediate from the Proposition. For the second, it suffices to note that the divisor in question has no components supported off $U_B^{m-k_x-k_y} \otimes_B U_B^{\lceil k_x+k_y \rceil}$. □

As an important consequence of this result, we can now determine the restriction of the G-bundles (i.e. the intermediate diagonals) on (essentially) the locus of cycles containing a node θ with given multiplicity; it is these restricted bundles that figure in the determination of the (polarized) node scrolls.

Proposition 7.5. *Let (T, δ, θ) be a boundary datum, $1 \leq j, n \leq m$ be integers, and consider the map*

$$\mu^n : (X^\theta)_T^{\lceil m-n \rceil} \to X_B^m$$
$$\mu^n(z) = c_{m-n}(z) + n\theta.$$

Then with $j_0 = \min(j, n)$, we have

(7.1.17)

$$(\mu^n)^*(OD_j^m(\theta)) \sim$$
$$-\binom{n-j_0+1}{2}\psi_x - \binom{j_0}{2}\psi_y + (n-j_0+1)\lceil m-n \rceil_* \theta_x + j_0 \lceil m-n \rceil_* \theta_y + \Gamma^{\lceil m-n \rceil}$$

where $\psi_x = \omega_{X_T^\theta/T} \otimes \mathcal{O}_{\theta_x}$ is the cotangent (psi) class at θ_x (which is a class from T, pulled back to $(X^\theta)_T^{\lceil m-n \rceil}$), $\lceil m-n \rceil_* \theta_x = \sum_{i=1}^{m-n} p_i^*(\theta_x)$ (which is a class from $(X^\theta)_T^{m-n}$, pulled back to $(X^\theta)_T^{\lceil m-n \rceil}$), and likewise for y.

Proof. Set $k = m - n$. We factor μ^n through the map

$$\mu_{j_0}^n : (X^\theta)_T^{\lceil k \rceil} \to (X^\theta)_T^m$$
$$\mu_{j_0}^n(z) = ((\theta_x)^{n-j_0+1}, (\theta_y)^{j_0-1}, c_k(z)).$$

We may write G_j^m as $G_{j_0}^n$ times a partial diagonal equation as above, and the inequalities on j_0 ensure that $G_{j_0}^n$ does not vanish identically on $\beta_x^{n-j_0+1} \times \beta_y^{j_0-1}$, where β_x, β_y are the branch neighborhoods of θ_x, θ_y in X_T^θ. Then it is straightforward that the last two summands in (7.1.16) correspond to the last three summands in (7.1.17), e.g. a diagonal $D_{a,b}^m$ with $a \leq n - j_0 + 1$ coincides with $p_b^* \theta_x$. So it's just a matter of evaluating the pullback of $OD_j^n(\theta)$. For the latter, we use a Laplace (block) expansion of G_{j_0} on the first $n - j_0 + 1$ rows. In this expansion, the leading term is the first, i.e. the product of the two corner blocks. There writing $x_a - x_b = dx$, a generator of ψ_x, and likewise for ψ_y, we get the asserted form as in (7.1.17). $\square$

In view of Proposition 7.5, it is natural to set

$$D_j^n(\theta) =$$
$$-\binom{n-j+1}{2}\psi_x - \binom{j}{2}\psi_y + (n-j+1)[m-n]_*\theta_x + j[m-n]_*\theta_y \tag{7.1.18}$$
$$\in \operatorname{Pic}(X_T^\theta)^{[m-n]}$$

where $[k]_* = \frac{1}{k!}\omega_{k*}\lceil k \rceil$ (i.e. the 'norm' of a line bundle on the symmetric product, which exists as line bundle, not just with $\mathbb{Q}$-coefficients). This may be considered as line bundle

or divisor class. We will use the same notation to denote the corresponding divisor on a family obtained from X^θ_T by base change and blowing up of sections disjoint from $\theta_x \cup \theta_y$.

7.2. **Polarized structure of node polyscrolls.** The following Lemma is critical in determining the structure of a node scroll:

Lemma 7.6. *Let $Q^{n,m}_j$ be the canonical section of type Q^n_j in the node scroll $F^n_j = F^n_j(\theta) \subset X^{[m]}_B$. Then up to linear equivalence, we have, where $k = m - n$,*

$$\Gamma^{(m)}.Q^{n,m}_j \sim -\binom{n-j+1}{2}\psi_x - \binom{j}{2}\psi_y + (n-j+1)[k]_*\theta_x + j[k]_*\theta_y + \Gamma^{(k)} \tag{7.2.1}$$

Proof. It suffices to prove the ordered version, that is

(7.2.2)
$$\Gamma^{\lceil m\rceil}.OQ^{n,m}_j \sim -\binom{n-j+1}{2}\psi_x - \binom{j}{2}\psi_y + (n-j+1)\sum_i p_i^*\theta_x + j\sum_i p_i^*\theta_y + \Gamma^{\lceil k\rceil}.$$

We note that the map $Q^{n,m}_j \to X^{[m]}_B$ is given by (6.2.3) (and similarly for the ordered analogue) We split $\Gamma^{\lceil m\rceil}$ as a sum of 3 (a priori Weil) divisors $\Gamma_{n]} + \Gamma^{\lceil k\rceil} + \Gamma_{\text{mixed}}$ corresponding to the diagonal on the first n coordinates, the last k, and the mixed pairs. Consider the pullback (restriction) of these divisors on $Q = OQ^{n,m}_j$. Clearly the last two divisors meet Q properly. In fact, as Q is contained in the flag-Hilbert scheme the second one coincides with the pullback of the polarization from $X^{\lceil k\rceil}_B$. As for the mixed divisor, i.e $\sum\limits_{i\leq n<r} D_{i,r}$, it clearly restricts on Q to a linear combination of $\sum\theta_{x,i}$ and $\sum\theta_{y,i}$. The coefficients can be determined locally, at a point of type $Q^{n+k+x+k+y}_{j+k_y}$ on $X^{[m]}_B$. Because near Q^n_j, $\Gamma^{\lceil m\rceil}$ is generated by G_j, the coefficients can be read off from (7.1.17) (where $j_0 = j$ since $j \leq n$).

Finally the first divisor, which comes from $\sum\limits_{i<r\leq n} D_{i,r}$ does not come from a proper intersection. However, we note that it is independent of the fibre coordinates of $(X^\theta)^{\lceil k\rceil}_T$, i.e. comes from a line bundle on T, and depends only on n and not on m. This bundle can be evaluated by looking in a neighborhood of $\theta_x \cup \theta_y$, as above, where G_j is a generator. Then again the local formula (7.1.17) serves to conclude. □

We are now in position to determine the 'polarized' structure of the node scroll F^n_j, i.e. a vector bundle E such that $F^n_j = \mathbb{P}(E)$ and such that the canonical $\mathcal{O}(1)$ polarization on $\mathbb{P}(E)$ corresponds to $-\Gamma^{(m)}$.

Theorem 7.7 (Node scroll theorem)**.** *For any boundary datum (T, δ, θ), and any $1 \leq j < n \leq m$, there is an isomorphism*

$$F^n_j(\theta) \simeq \mathbb{P}(\mathcal{O}(D^n_j(\theta)) \oplus \mathcal{O}(D^n_{j+1}(\theta))) \tag{7.2.3}$$

under which the canonical $\mathcal{O}(1)$ polarization on a projectivization corresponds to the restriction of $-p^*_{X_B^{[m]}}\Gamma^{(m)} + p^*_{(X_T^{\theta.})^{[m-n]}}\Gamma^{(m-n)}$.

Proof. As $F_j^n(\theta)$ admits the two disjoint sections $Q_j^{n,m}, Q_{j+1}^{n,m}$, the result is immediate from Lemma 7.6. □

Corollary 7.8. *On $F_j^{n,m}(\theta)$, we have*

$$\begin{aligned} -\Gamma^{(m)} &\sim Q_j^{n,m} + p^*_{[m-n]}(D^n_{j+1}) \\ &\sim Q_{j+1}^{n,m} + p^*_{[m-n]}(D^n_j). \end{aligned} \tag{7.2.4}$$

Proof. Follows from the fact that on any $\mathbb{P}^1$-bundle $\mathbb{P}(A \oplus B)$ with projection π, we have

$$\mathcal{O}(1) \sim \mathbb{P}(A) \oplus \pi^*(B).$$

□

The extension to polyscrolls is direct from the definition in §6.3 once we note that thanks to the disjointness of the nodes, the divisors $D_{j_i}^{n_i}(\theta_i)$ correspond naturally to a similarly-denoted divisor on $(X_T^{\theta.})^{[m-|n.|]}$, with $[m-n_i]_*\theta_{x,i}$ corresponding to $[m-|n.|]_*\theta_{x,i}$, e.g. on $F_{j_1}^{n_1}(\theta_1)$,

$$p^*_{[m]}[m]_*\theta_{2,x} = p^*_{[m-n_1]}[m-n_1]_*\theta_{2,x}$$

etc. where $p_{[k]}$ denotes the natural map to the length-k Hilbert scheme (of X or X^{θ_1}) (compare Lemma 6.3).

Theorem 7.9 (Node Polyscroll Theorem)**.** *There is an isomorphism*

$$F_{j.}^{n.}(\theta.; X/B) \sim \prod_{(X^{\theta.}_{T(\theta.)})^{[m-|n.|]}} \mathbb{P}(\mathcal{O}(D_{j_i}^{n_i}(\theta_i)) \oplus \mathcal{O}(D_{j_i+1}^{n_i}(\theta_i))) \tag{7.2.5}$$

under which $-\Gamma^{(m)} + \Gamma^{(m-|n.|)}$ corresponds to the canonical $\mathcal{O}(1, ..., 1)$.

Proof. We use the setting and notations of §6.3. Consider the natural projection $F \to F'$, which is just a base-change of the scroll projection

$$p_{[m-n_1]} : F_1 = F_{j_1}^{n_1,m}(\theta_1, X/B) \to (X^{\theta_1})^{[m-n_1]}.$$

Via this, we have

$$\mathcal{O}_{F_1}(1) = \binom{n_1 - j_1 + 1}{2}\psi_{1,x} + \binom{j_1}{2}\psi_{1,y} - (n_1 - j_1 + 1)[m-n_1]_*(\theta_{1,x}) - j_1[m-n_1]_*\theta_{1,y}.$$

On F this becomes, using Lemma 6.3 (essentially, the disjointness of the sections θ_i),

$$\mathcal{O}_{F_1}(1)|_F = \binom{n_1 - j_1 + 1}{2}\psi_{1,x} + \binom{j_1}{2}\psi_{1,y} - (n_1 - j_1 + 1)[m-|n.|]_*(\theta_{1,x}) - j_1[m-|n.|]_*(\theta_{1,y}).$$

and by Theorem 7.7, this coincides on F with $-\Gamma^{(m)} + \Gamma^{(m-n_1)}|_{F'}$. By induction, $-\Gamma^{(m-n_1)}|_{F'} + \Gamma^{(m-|n.|)}$ coincides with the appropriate $\mathcal{O}(1, ..., 1)$ on the $(r-1)$-polyscroll F', and the Theorem follows.

□

Remark 7.10. Define a *smudgy curve* of type g, p, k to be a stable p-pointed curve together with a length-k subscheme, and let $\overline{\mathcal{M}}^{[k]}_{g,p}$ denote the moduli space (stack) of smudgy curves of this type (assuming it exists). Some interesting questions about (ordinary) curves (for example, Brill-Noether loci) can be formulated in terms of smudgy curves. The node scrolls define correspondences between smudgy moduli spaces:

$$\overline{\mathcal{M}}^{[k_1]}_{g_1,p_1+1} \times \overline{\mathcal{M}}^{[k_2]}_{g_2,p_2+1} \leftarrow \quad F^n_j \quad \to \overline{\mathcal{M}}^{[k_1+k_2+n]}_{g_1+g_2,p_1+p_2},$$
$$\overline{\mathcal{M}}^{[k]}_{g-1,p+2} \leftarrow \quad F^n_j \quad \to \overline{\mathcal{M}}^{[k+n]}_{g,p}$$

(identifying the LHS with a boundary component of $\overline{\mathcal{M}}^{[k]}_{g,p}$). These are analogous to the correspondences used by Nakajima [2] to define creation-annihilation operators on the cohomology of Hilbert schemes of surfaces. We plan to explore this aspect further in future work [5].

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